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A Note on the Computational Stability of the Two-Step Lax-Wendroff Form of the Advection Equation

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ABSTRACT—It is shown by elementary analysis that the two-step Lax-Wendroff method for integrating the advection equation is subject to nonlinear computational instability. The source of the instability lies in the possibility of lattice separation in the field of the advecting coefficient.

1. INTRODUCTION

The use of centered finite-difference approximations for the quasi-linear system of equations that governs the physical behavior of inviscid quasi-static models of the atmosphere is known to possess deficiencies with respect to the stability of the computations. These deficiencies have been analyzed by Phillips (1959), Richtmyer (1963) and, recently, by Robert et al. (1970).

In the paper by Robert et al., a heuristic analysis, motivated by some remarks in Richtmyer's (1963) paper, was employed. In essence, the analysis is based on the representation of the high frequency or wave number content of a field by its "folded" low frequencies or wave number equivalent, modulated by a two grid-interval oscillation. If the folded representation of a field is of sufficiently low frequency or wave number, then, following Richtmyer, one may treat the field as a relative constant in the stability analysis of a quasi-linear system of equations. The simplest such system is a one space-dimension advection equation.

In the present note, we apply the analysis method to an advection equation formulated using the two-step Lax-Wendroff method (Richtmyer and Morton 1967). It is known from the work of several researchers (see Houghton et al. 1966) that the two-step, Lax-Wendroff method does permit the development of nonlinear instability, and that the method permits the separation of solutions on alternate lattices. For these reasons, the results of the analysis are readily anticipated. This presentation is offered to show the connection between the "damping scheme" here analyzed and the "neutral scheme" analyzed in Robert et al. (1970). Attention is focused on lattice separation as the causative factor for nonlinear instability in centered-difference schemes.

2. ANALYSIS

Consider the advection equation,

$$\frac{\partial V}{\partial t}=-A\,\frac{\partial V}{\partial x}, \qquad (1)$$

in which the field, V, is advected by the field, A. Generally, the field A will be a function of time, t, and space, x. The two-step Lax-Wendroff scheme for eq (1) follows:

Step 1:

$$V_j^{n+1}=\frac{1}{2}\,[V_{j+1}^n+V_{j-1}^n]-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)A_j^n[V_{j+1}^n-V_{j-1}^n]. \qquad (2)$$

Step 2:

$$V_j^{n+2}=V_j^n-\left(\frac{\Delta t}{\Delta x}\right)A_j^{n+1}[V_{j+1}^{n+1}-V_{j-1}^{n+1}]. \qquad (3)$$

The superscripts and subscripts are integer indices indicating the time and space levels at which the parameters are evaluated. Δt and Δx are the time and space increments.

Allowing for the variability of A, one may combine eq (1) and (2) into a single equation:

$$V_j^{n+2}=V_j^n-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)A_j^{n+1}[V_{j+2}^n-V_{j-2}^n]+\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2[A_j^{n+1}A_{j+1}^n(V_{j+2}^n-V_j^n)-A_j^{n+1}A_{j-1}^n(V_j^n-V_{j-2}^n)]. \qquad (4)$$

We now assume, following the method of Robert et al. (1970), that the field A may be represented by a series of elements of the form

$$A_j^n=[A_0^n+A_1^n e^{i\pi j}]e^{ipj\Delta x}. \qquad (5)$$

It will be further assumed that the wave number p is sufficiently small so that the trigonometric function is essentially constant. Thus, we write

$$A_j^n=[A_0^n+A_1^n e^{i\pi j}]. \qquad (6)$$

It is to be understood that A_1^n represents the amplitude of the high wave number element which is the folded counterpart of the low wave number p. We shall also assume that the temporal variability of A_j^n is small, and therefore introduce

$$A_j^n = A_0 + A_1 e^{i\pi j} \tag{7}$$

into eq (4). One obtains the equation

$$V_j^{n+2} = V_j^n - \frac{1}{2}\left[\frac{\Delta t}{\Delta x}\right][A_0 + A_1 e^{i\pi j}][V_{j+2}^n - V_{j-2}^n] + \frac{1}{2}\left[\frac{\Delta t}{\Delta x}\right]^2 [A_0^2 - A_1^2][V_{j+2}^n - 2V_j^n + V_{j-2}^n]. \tag{8}$$

A stability analysis of eq (8) may be performed by first noting that only even or only odd values of the index, j, appear in the equation. We shall consider j even and look for solutions of the form

$$V_j^n = \zeta^n e^{ikj\Delta x} \tag{9}$$

in which

$$0 \leq k\Delta x \leq \frac{\pi}{2} \tag{10}$$

and ζ is a complex constant. We must note that the neglect of the trigonometric dependence of A [eq (5) and (6)] implies that our analysis is valid only for sufficiently large values of $k\Delta x$ in eq (10). Note that the wave number for which $k\Delta x$ has its maximum value corresponds to a $4\Delta x$ wavelength component of V.

Upon substitution of eq (9) into eq (8), one obtains the following relation to be satisfied by ζ:

$$|\zeta^2|^2 = 1 + 4\epsilon^2 \sin^2 k\Delta x[2\gamma(1+\gamma)] + 4\epsilon^2 \sin^4 k\Delta x[\epsilon^2(1-\gamma^2)^2 - (1+\gamma)^2] \tag{11}$$

in which we have used the definitions

$$\epsilon = \frac{A_0 \Delta t}{\Delta x} \text{ and } \gamma = \frac{A_1}{A_0}. \tag{12}$$

If the solution [eq (9)] is to remain bounded as n increases, it is necessary that

$$|\zeta| < 1. \tag{13}$$

If eq (13) holds, then one also has

$$|\zeta^2|^2 < 1. \tag{14}$$

Thus, a necessary condition for stability is

$$\epsilon^2 < \frac{(1+\gamma)^2}{(1-\gamma^2)^2}\left[1 - \frac{2\gamma}{(\sin^2 k\Delta x)\ (1+\gamma)}\right]. \tag{15}$$

Now when $A_1 = 0$, $\gamma = 0$, and eq (15) yields the "linear" stability criterion,

$$\frac{A_0 \Delta t}{\Delta x} < 1. \tag{16}$$

On the contrary, if $A_1 \neq 0$, then eq (15) must be satisfied. In particular, eq (15) indicates that it is necessary for

$$\left(\frac{\gamma}{1+\gamma}\right) < \frac{1}{2}\sin^2 k\Delta x \tag{17a}$$

or

$$A_1 < A_0\left[\frac{\sin^2 k\Delta x}{1+\cos^2 k\Delta x}\right]. \tag{17b}$$

If eq (17b) is not satisfied, then the solution will be unstable, *no matter how small* Δt *is made*. Consequently, violation of eq (17b) will lead to what has been called "nonlinear" instability. From eq (17b) and our interpretation of A_0 and A_1, it follows that the cause of the instability is the excessive degree of lattice separation.

3. CONCLUDING REMARKS

It seems clear that failure to control the lattice separability of centered-difference approximations of the quasi-linear meteorological equations is the essential cause of nonlinear instability. This point was made by Phillips (1959) but was obscured, at least in the mind of the writer, by Phillips' suggestion that aliased nonlinear interactions were the culprits.

Since lattice separation is not caused solely by nonlinear interactions but also by faulty boundary condition formulation or by highly variable forcing terms (orographic or diabatic), it is perhaps easier to understand the problem if one centers attention on the linear lattice structure of the numerical solutions. The presence of the nonlinear terms in the system of equations seems to require that solutions on different linear lattices be used in estimating these terms. Should the solutions on the other lattices differ appreciably, the problem analyzed in this paper will be a source of computational difficulty.

Finally, it should be noted that when nonlinear instability of the type analyzed here is encountered, one is hard pressed to justify continuation of the calculation from the viewpoint of weather prediction accuracy. In essence, the problem only arises when two, or more, large-scale numerical solutions that are uniquely defined on separate but equal lattices and with equal claim to legitimacy are nonetheless significantly different.

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